Voevodsky's contribution to the foundation of mathematics

Daniel R. Grayson

School on Univalent Mathematics - Cortona - July 17-23, 2022

Abstract

Homotopy type theory, with the partition of types into levels and the univalence axiom developed by Voevodsky, provides both a new logical foundation for mathematics (Univalent Foundations) and a formal language usable with computers for checking the proofs mathematicians make daily. As a foundation, it replaces set theory with a framework where propositions and sets are defined in terms of a more primitive notion called *type* – in this framework the notion of symmetry arises at the most basic level: from the logic. As a formal language, it encodes the axioms of mathematics and the rules of logic simultaneously, and promises to make the extraction of algorithms and values from constructive proofs easy. As a mathematical topic, it offers an intriguing range of open problems at all levels of accessibility.

I will give an intuitive introduction to these recent developments.

The memorial web site

www.math.ias.edu/Voevodsky/

Vladimir Voevodsky

Владимир Александрович Воеводский

4 June 1966 – 30 September 2017

Vladimir Voevodsky was an algebraist with a deep understanding of topology who found novel ways to apply topology to algebraic geometry and to the foundations and formalization of mathematics. His work on foundations was interrupted by his sudden death in September, 2017. This web site commemorates his work and his life and serves as an archive of his works, both complete and incomplete, for those wishing to examine and to extend his work.

Annals of Mathematics **174** (2011), 401–438
doi: 10.4007/annals.2011.174.1.11

On motivic cohomology with $\mathbf{Z}/l$-coefficients

By Vladimir Voevodsky

Abstract

In this paper we prove the conjecture of Bloch and Kato which relates Milnor's K-theory of a field with its Galois cohomology as well as the related comparisons results for motivic cohomology with finite coefficients in the Nisnevich and étale topologies.

The Fields Medal is awarded, 2002

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Tuesday, 20 August, 2002, 17:27 GMT 18:27 UK

Prize for 'big picture' mathematicians

By Richard Black
BBC Science Correspondent

Mathematicians from France and the United States have been awarded the world's top maths prize, the Fields Medal.

The medal is given every four years, and is regarded as the maths equivalent of the Nobel Prizes.

See also:

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From the laudatory article in the conference proceedings

ICM 2002 · Vol. I · 99–103

The Work of Vladimir Voevodsky

Christophe Soulé*

Vladimir Voevodsky is an amazing mathematician. He has demonstrated an exceptional talent for creating new abstract theories, about which he proved highly nontrivial theorems. He was able to use these theories to solve several of the main long standing problems in algebraic K-theory. The field is completely different after his work. He opened large new avenues and, to use the same word as Laumon, he is leading us closer to the world of *motives* that Grothendieck was dreaming about in the sixties.

From a public lecture, March 26, 2014

In 1999/2000, again at the IAS, I was giving a series of lectures, and Pierre Deligne was taking notes and checking every step of my arguments. Only then did I discover that the proof of a key lemma in "***Cohomological Theory***" contained a mistake and that the lemma, as stated, could not be salvaged.

Fortunately, I was able to prove a weaker and more complicated lemma which turned out to be sufficient for all applications. A corrected sequence of arguments was published in 2006.

A comparison

1994 version:

Proposition 4.22 *Let W be a smooth semi-local scheme over a field k and $W = U \cup V$ be an open covering of W. Then for any pretheory (F, ϕ, A) over k such that A is the category of abelien groups the following sequence is exact:*

$$0 \longrightarrow F(W) \longrightarrow F(U) \oplus F(V) \longrightarrow F(U \cap V) \longrightarrow 0.$$

2006 version:

LEMMA 22.10. *Suppose that F is a homotopy invariant presheaf with transfers. Then for any open covering $S = U_0 \cup V$ there is an open $U \subset U_0$ such that $S = U \cup V$ and the sequence $F(MV(Q))$ is exact, where $Q = Q(S, U, V)$:*

$$0 \to F(S) \to F(U) \oplus F(V) \to F(U \cap V) \to 0.$$

From a public lecture, March 26, 2014

This story got me scared. Starting from 1993 multiple groups of mathematicians studied the "***Cohomological Theory***" paper at seminars and used it in their work and none of them noticed the mistake.

And it clearly was not an accident. ***A technical argument by a trusted author, which is hard to check and looks similar to arguments known to be correct, is hardly ever checked in detail.***

An email from 2002

```
Date:  Tue, 10 Sep 2002 09:15:21 -0400 (EDT)
From:  Vladimir Voevodsky <vladimir@ias.edu>
To:  dan@math.uiuc.edu

...

Vladimir.

PS I am thinking again about the applications of computers to pure
math.  Do you know of anyone working in this area?  I mean mostly
some kind of a computer language to describe mathematical structures,
their properties and proofs in such a way that ultimately one may
have mathematical knowledge archived and logically verified in a
fixed format.
```

A very short note on homotopy λ-calculus

Vladimir Voevodsky

September 27, 2006

The homotopy λ-calculus is a hypothetical (at the moment) type system. To some extent one may say that $H\lambda$ is an attempt to bridge the gap between the "classical" type systems such as the ones of PVS or HOL Light and polymorphic type systems such as the one of *Coq*. The main problem with the polymorphic type systems lies in the properties of the equality types. As soon as we have a universe U of which $Prop$ is a member we are in trouble. In the Boolean case, $Prop$ has an automorphism of order 2 (the negation) and it is clear that this automorphism should correspond to a member of $Eq(U; Prop, Prop)$. However, as far as I understand there is no way to produce such a member in, say, *Coq*. A related problem looks as follows. Suppose $T, T' : U$ are two type expressions and there exists an isomorphism $T \to T'$ (the later notion of course requires the notion of equality for members of T and T'). Clearly, any proposition which is true for T should be true for T' i.e. for all functions $P : U \to Prop$ one should have $P(T) = P(T')$. Again as far as I understand this can not be proved in *Coq* no matter what notion of equality for members of T and T' we use.

Here is the general picture as I understand it at the moment. Let us consider the type system TS which is generated by the sequents

$$\vdash U_i : U_{i+1}$$

$2 \in 4$?

$$2 \in 4\ ?$$

If $4 := \{0, 1, 2, 3\}$, then yes.

$2 \in 4$?

If $4 := \{0, 1, 2, 3\}$, then yes.

If $4 := \{3\}$, then no.

$$2 \in 4\ ?$$

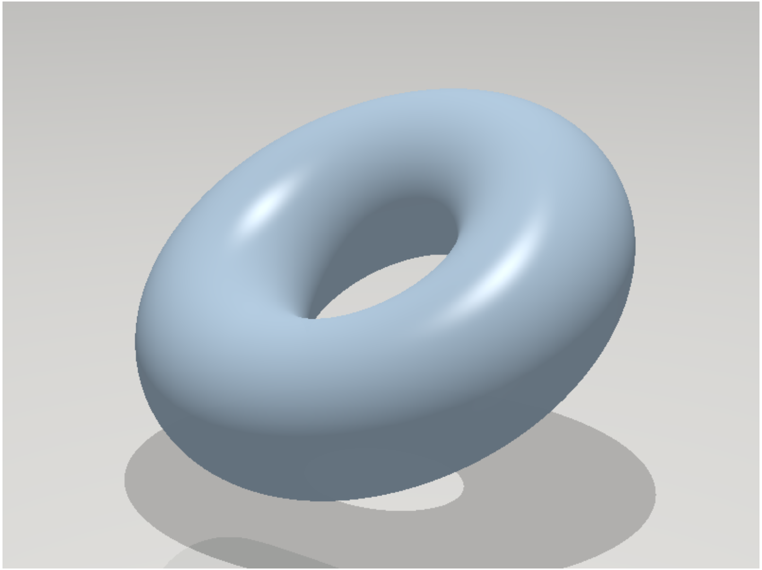
If $4 := \{0, 1, 2, 3\}$, then yes.

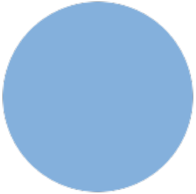
If $4 := \{3\}$, then no.

The two approaches yield isomorphic models of the natural numbers, so the answer ought to be the same, according to Voevodsky's insight.

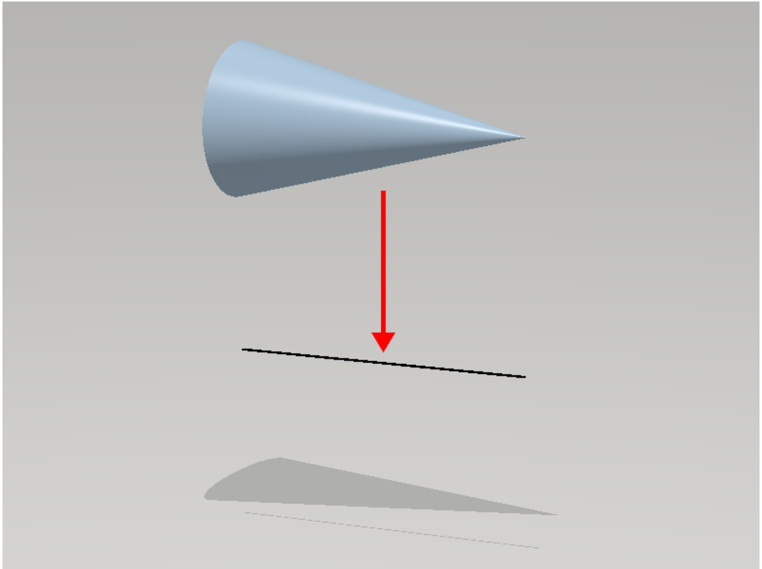
Introduction to topology

The animations on this slide and on subsequent slides can be viewed with Acrobat Reader.

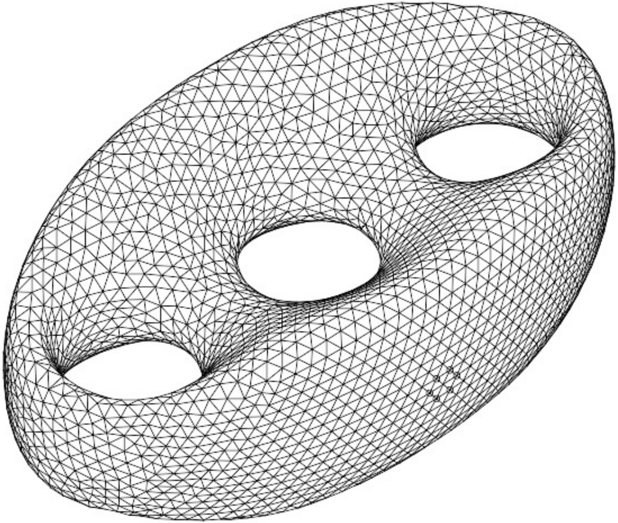




A fibration



Topology in combinatorial style



The syntax of Martin-Löf type theory

$$x : X$$

$$f : X \to Y$$

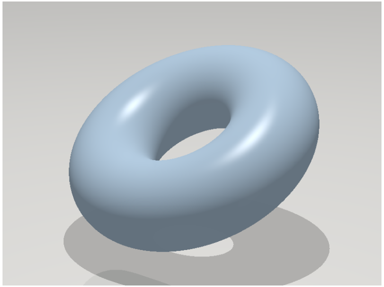
$$p : x = x'$$

$$q : X = Y$$

$$X, Y : U$$

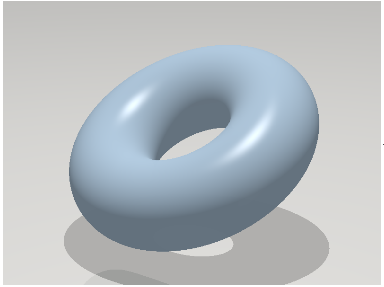
$$f : X \cong Y$$

Univalence Axiom: $(X = Y) \xrightarrow{\cong} (X \cong Y)$

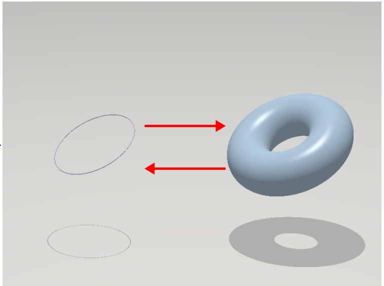


Univalence Axiom: $(X = Y) \xrightarrow{\cong} (X \cong Y)$

…as viewed in topology:



$\xrightarrow{\cong}$



Étale Cohomology

J. S. Milne

An injection is denoted by $\hookrightarrow$, a surjection by $\twoheadrightarrow$, an isomorphism by $\approx$, a quasi-isomorphism (or homotopy) by $\sim$, and a canonical isomorphism by $=$. The symbol $X \overset{df}{=} Y$ means X is defined to be Y, or that X equals Y by definition.

Date: Mon, 01 May 2006 10:10:30 CDT
To: Peter May <may@math.uchicago.edu>
From: "A. Bousfield" <bous@uic.edu>
Subject: Re: Simplicial question

Dear Peter,

I think that the answer to Voevodsky's basic question is "yes," and I'll try to sketch a proof.

Since the Kan complexes X and Y are homotopy equivalent, they share the same minimal complex M, and we have trivial fibrations X -> M and Y -> M by Quillen's main lemma in "The geometric realization of a Kan fibration ." Thus X + Y -> M + M is also a trivial fibration where "+" gives the disjoint union. We claim that the composition of X + Y -> M + M with the inclusion M + M >-> M x Delta^1 may be factored as the composition of an inclusion X + Y >-> E with a trivial fibration E -> M x Delta^1 such that the counterimage of M + M is X + Y. We may then obtain the desired fibration

E -> M x Delta^1 -> Delta^1

whose fiber over 0 is X and whose fiber over 1 is Y.

We have used a case of:

Claim. The composition of a trivial fibration A -> B with an inclusion B -> C may be factored as the composition of an inclusion A >-> E with a trivial fibration E -> C such that the counterimage of B is A.

...

His first lecture about univalent foundations

The equivalence axiom and univalent models of type theory.

(Talk at CMU on February 4, 2010)

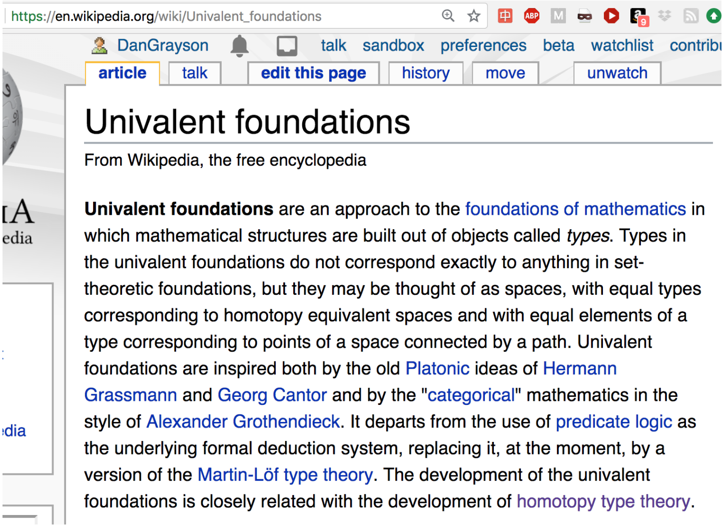
By Vladimir Voevodsky

Abstract

I will show how to define, in any type system with dependent sums, products and Martin-Lof identity types, the notion of a homotopy equivalence between two types and how to formulate the Equivalence Axiom which provides a natural way to assert that "two homotopy equivalent types are equal". I will then sketch a construction of a model of one of the standard Martin-Lof type theories which satisfies the equivalence axiom and the excluded middle thus proving that M.L. type theory with excluded middle and equivalence axiom is at least as consistent as ZFC theory.

Models which satisfy the equivalence axiom are called univalent. This is a totally new class of models and I will argue that the semantics which they provide leads to the first satisfactory approach to type-theoretic formalization of mathematics.

Univalent Foundations



https://en.wikipedia.org/wiki/Univalent_foundations

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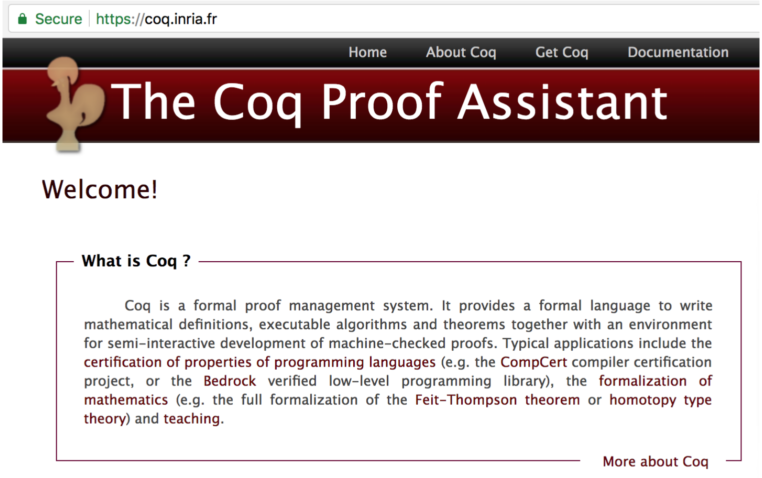
Univalent foundations

From Wikipedia, the free encyclopedia

Univalent foundations are an approach to the foundations of mathematics in which mathematical structures are built out of objects called *types*. Types in the univalent foundations do not correspond exactly to anything in set-theoretic foundations, but they may be thought of as spaces, with equal types corresponding to homotopy equivalent spaces and with equal elements of a type corresponding to points of a space connected by a path. Univalent foundations are inspired both by the old Platonic ideas of Hermann Grassmann and Georg Cantor and by the "categorical" mathematics in the style of Alexander Grothendieck. It departs from the use of predicate logic as the underlying formal deduction system, replacing it, at the moment, by a version of the Martin-Löf type theory. The development of the univalent foundations is closely related with the development of homotopy type theory.

Coq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The Coq Proof Assistant

Welcome!

What is Coq ?

Coq is a formal proof management system. It provides a formal language to write mathematical definitions, executable algorithms and theorems together with an environment for semi-interactive development of machine-checked proofs. Typical applications include the certification of properties of programming languages (e.g. the CompCert compiler certification project, or the Bedrock verified low-level programming library), the formalization of mathematics (e.g. the full formalization of the Feit-Thompson theorem or homotopy type theory) and teaching.

More about Coq

The notion of h-level, in his *Foundations*

GitHub, Inc. [US] | https://github.com/UniMath/Foundations/blob/master/Generalities/uu0.v

```
(** *** h-levels of types *)


Fixpoint isofhlevel (n:nat) (X:UU): UU:=
match n with
O => iscontr X |
S m => forall x:X, forall x':X, (isofhlevel m (paths x x'))
end.
```

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A type X is of *h-level* 0 if it has a unique element.

A type X is of *h-level* $\leq n+1$ if for all x, y in X, the type $x = y$ is of h-level $\leq n$.

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. . . alternatively: . . .

A type X is of *h-level* 1 if any two elements of it are equal.
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...and then we define: ...

A type X is a *proposition* if it is of h-level ≤ 1 [Awodey-Bauer, 2001].
A type X is a *set* if it is of h-level ≤ 2.

Voevodsky's notion of h-level

h-level	*type* T	*elements*	*identity types*
0	*true*		
1	*proposition*	proofs p, q	$p = q$ is true

Voevodsky's notion of h-level

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2	*set*	elements x, y	$x = y$ is a proposition
3	*a type of h-level* ≤ 3	elements G, H	$G = H$ is a set

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$\vdots$	$\vdots$	$\vdots$	$\vdots$

The univalence axiom, in his *Foundations*

GitHub, Inc. [US] | https://github.com/UniMath/Foundations/blob/master/Proof_of_Extensionality/funextfun.v

```
(** ** Univalence axiom. *)


Definition eqweqmap { T1 T2 : UU } ( e: paths T1 T2 ) : weq T1 T2 .
Proof. intros. destruct e . apply idweq. Defined.

Axiom univalenceaxiom :  forall T1 T2 : UU ,  isweq ( @eqweqmap T1 T2 ).
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GitHub, Inc. [US] | https://github.com/UniMath/Foundations/blob/master/Proof_of_Extensionality/funextfun.v

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A function $f : X \to Y$ is an *equivalence* if, for each y in Y, there is just one x in X with $f(x) = y$.

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A function $f : X \to Y$ is an *equivalence* if, for each y in Y, there is just one x in X with $f(x) = y$.

The notation for a function being an equivalence is $f : X \xrightarrow{\cong} Y$.
The notation for the type of all equivalences between X and Y is $X \cong Y$.

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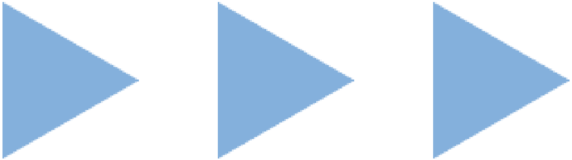
$$\textit{Univalence Axiom:}\ (X = Y) \xrightarrow{\cong} (X \cong Y)$$

An example of a type of h-level 3

The type of all triangles (with unlabeled vertices).



Feasibility of the encoding, in his *Foundations*

- functions whose corresponding values are all equal, are equal
- the type of functions from one set to another is a set
- a subtype of a set is a set (call it a *subset*)
- the type of all subsets* of a set is a set;
- whether a type is of h-level $\leq n$, is a proposition
- equivalences have inverse functions that are equivalences
- whether a function is an equivalence, is a proposition
- the type of natural numbers and the finite types are sets
- equivalent types have the same h-level
- propositions that imply each other are equivalent
- subsets defined by equivalent predicates are equal

How to encode the notion of "group"

Let U be a *universe*.

A *group* in U is a sequence $(G, e, i, m, \lambda, \rho, \lambda', \rho', \alpha, \iota)$, where

- G is a type of U
- $e : G$
- $i : G \to G$
- $m : G \times G \to G$
- λ is a proof that for every $a : G$, $m(e, a) = a$
- ρ is a proof that for every $a : G$, $m(a, e) = a$
- λ' is a proof that for every $a : G$, $m(i(a), a) = e$
- ρ' is a proof that for every $a : G$, $m(a, i(a)) = e$
- α is a proof that for every $a, b, c : G$, $m(m(a, b), c) = m(a, m(b, c))$
- ι is a proof that G is a set

Another way to encode the notion of “group”

Let U be a *universe*.

A *group* in U is a sequence (BG, w, c, ℓ), where

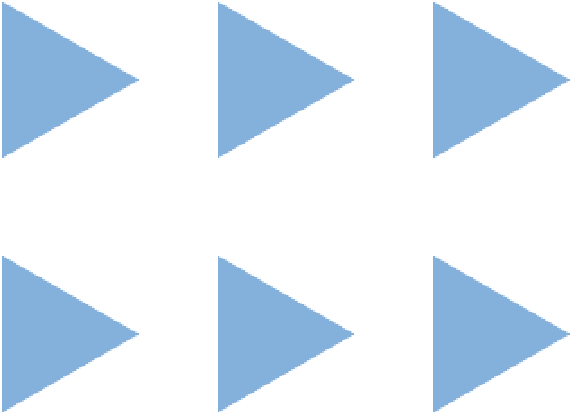
- BG is a type of U
- $w : BG$
- c is a proof that BG is *connected*, i.e., $\forall x : BG\ \exists p : x = w$
- ℓ is a proof that BG is of h-level ≤ 3

A *group homomorphism* from (BG, w, c, ℓ) to (BG', w', c', ℓ') is

- a function $f : BG \to BG'$
- an identity $p : f(w) = w'$

An example of a group of order 6

The type of all triangles having all sides of length 1,
with one triangle chosen (for movies to start and end with).



The special year

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Univalent Foundations of Mathematics

Monday, September 24, 2012 (All day) to Thursday, August 15, 2013 (All day)

2012-2013

The book

Homotopy Type Theory

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The HoTT Book

Homotopy Type Theory:
Univalent Foundations of Mathematics
The Univalent Foundations Program
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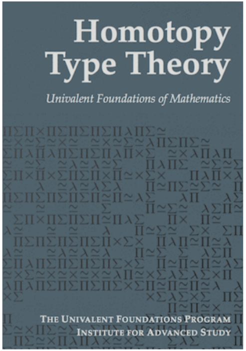

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The UniMath repository of proofs

GitHub, Inc. [US] | https://github.com/UniMath/UniMath

Univalent Mathematics

This Coq library aims to formalize a substantial body of mathematics using the univalent point of view.

Installation

See INSTALL.md.

Contents

The UniMath subdirectory contains various packages of formalized mathematics. For more information, see the UniMath Table of Contents.

Some scientific articles describing the contents of the UniMath library are listed in the wiki.

Contributing to UniMath

To contribute to UniMath, submit a pull request. Your code will be subject to the copyright and license agreement in LICENSE.md.

For the style guide and other instructions, see UniMath/README.md.

Vladimir's final project

MODELS, INTERPRETATIONS AND THE INITIALLITY CONJECTURES

VLADIMIR VOEVODSKY

Abstract. Work on proving consistency of the intensional Martin-Löf type theory with a sequence of univalent universes ("MLTT+UA") led to the understanding that in type theory we do not know how to construct an interpretation of syntax from a model of inference rules. That is, we now have the concept of a model of inference rules and the concept of an interpretation of the syntax and a conjecture that implies that the former always defines the latter. This conjecture, stated as the statement that the term model is an initial object in the category of all models of a given kind, is called the Initiallity Conjecture. In my talk I will outline the various parts of this new vision of the theory of syntax and semantics of dependent type theories.

1. Introduction

The first few steps in all approaches to the set-theoretic semantics of dependent type theories remain insufficiently understood. The constructions which have been worked out in detail in the case of a few particular type systems by dedicated authors are being extended to the wide variety of type systems under consideration today by analogy. This is not acceptable in mathematics. Instead we should be able to obtain the required results for new type systems by *specialization* of general theorems and constructions formulated for abstract objects the instances of which combine together to produce a given type system.

C-system of a module over a Jf-relative monad

Vladimir Voevodsky

(Submitted on 1 Feb 2016)

Let F be the category with the set of objects $\mathbf{N}$ and morphisms being the functions between the standard finite sets of the corresponding cardinalities. Let $Jf : F \to Sets$ be the obvious functor from this category to the category of sets. In this paper we construct, for any relative monad $\mathbf{RR}$ on Jf and a left module $\mathbf{LM}$ over $\mathbf{RR}$, a C-system $C(\mathbf{RR}, \mathbf{LM})$ and explicitly compute the action of the B-system operations on its B-sets.
In the following paper it is used to provide a rigorous mathematical approach to the construction of the C-systems underlying the term models of a wide class of dependent type theories.
This paper is a result of evolution of arXiv:1407.3394. However this paper is much more detailed and contains a lot of material that is not contained in arXiv:1407.3394. It also does not cover some material that is covered in arXiv:1407.3394.

arXiv > math > arXiv:1505.06446

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Mathematics > Category Theory

[Submitted on 24 May 2015]

Martin-Lof identity types in the C-systems defined by a universe category

Vladimir Voevodsky

This paper continues the series of papers that develop a new approach to syntax and semantics of dependent type theories. Here we study the interpretation of the rules of the identity types in the intensional Martin-Lof type theories on the C-systems that arise from universe categories. In the first part of the paper we develop constructions that produce interpretations of these rules from certain structures on universe categories while in the second we study the functoriality of these constructions with respect to functors of universe categories. The results of the first part of the paper play a crucial role in the construction of the univalent model of type theory in simplicial sets.

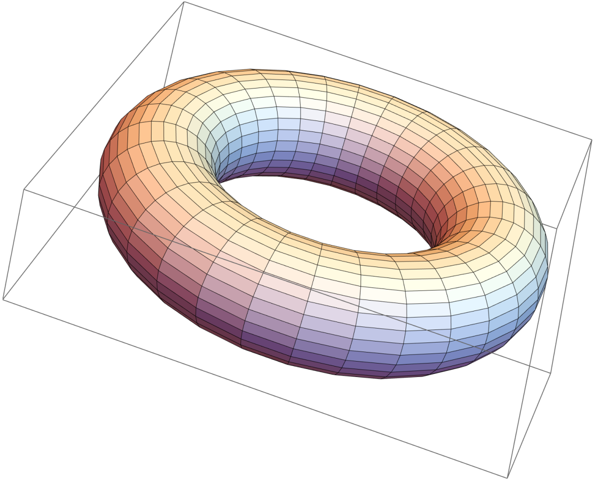
For (3.47), we proceed as follows.

$$
\begin{aligned}
&\psi_\Gamma \circ \Phi(\phi(\Gamma, T, P, s0)) \circ R_\Phi \circ \mathsf{pr}_2' \\
&\quad = \psi_\Gamma \circ \Phi(\phi(\Gamma, T, P, s0)) \circ \Phi(\mathsf{pr}_2) \circ \widetilde{\xi}_\Phi && \text{(by 3.41)} \\
&\quad = \psi_\Gamma \circ \Phi(\phi(\Gamma, T, P, s0) \circ \mathsf{pr}_2) \circ \widetilde{\xi}_\Phi && \text{(by functoriality of } \Phi) \\
&\quad = \psi_\Gamma \circ \Phi(\eta^{!}{}_{p}^{-1}((F, \widetilde{H}))) \circ \widetilde{\xi}_\Phi && \text{(by 3.43)} \\
&\quad = \psi_\Gamma \circ \Phi(\eta^{!}{}_{p}^{-1}((u_1(T), \widetilde{u}_1(s0))) \circ \widetilde{\xi}_\Phi && \text{(by def'n of } F \text{ and } \widetilde{H}) \\
&\quad = \psi_\Gamma \circ \Phi(\eta^{!}{}_{p}^{-1}((u_1(\mathsf{ft}(\partial(s0))), \widetilde{u}_1(s0))) \circ \widetilde{\xi}_\Phi && \text{(by the type of } s0) \\
&\quad = \psi_\Gamma \circ \Phi(\eta^{!}{}_{p}^{-1}(\widetilde{u}_{2,\Gamma}(s0))) \circ \widetilde{\xi}_\Phi && \text{(by def'n of } \widetilde{u}_2 \text{ in 3.24)} \\
&\quad = \eta^{!}{}_{p'}^{-1}(\widetilde{u}'_{2,H(\Gamma)}(H(s0))) && \text{(by [15, Lemma 6.2(2)])} \\
&\quad = \eta^{!}{}_{p'}^{-1}((u_1'(\mathsf{ft}(\partial(H(s0)))), \widetilde{u}_1'(H(s0)))) && \text{(by def'n of } \widetilde{u}_2 \text{ in 3.24)} \\
&\quad = \eta^{!}{}_{p'}^{-1}((u_1'(\mathsf{ft}(\partial(H(s0)))), \widetilde{H}')) && \text{(by def'n of } \widetilde{H}') \\
&\quad = \eta^{!}{}_{p'}^{-1}((u_1'(\mathsf{ft}(H(\partial(s0)))), \widetilde{H}')) && \text{(by functoriality of } H) \\
&\quad = \eta^{!}{}_{p'}^{-1}((u_1'(H(\mathsf{ft}(\partial(s0)))), \widetilde{H}')) && \text{(by 3.19)} \\
&\quad = \eta^{!}{}_{p'}^{-1}((u_1'(H(T))), \widetilde{H}')) && \text{(by the type of } s0) \\
&\quad = \eta^{!}{}_{p'}^{-1}((F', \widetilde{H}')) && \text{(by def'n of } F') \\
&\quad = \phi(H(\Gamma), H(T), H(P), H(s0)) \circ \mathsf{pr}_2' && \text{(by 3.45)}
\end{aligned}
$$

Topology in another combinatorial style

In[134]:=
```
n = 30; d = 2 Pi / n;
g[i_, j_] := 3 * {Cos[i], Sin[i], 0} + {Cos[i] Cos[j], Sin[i] Cos[j], Sin[j]};
f[i_, j_] := Polygon[ {g[i, j], g[i + d, j], g[i + d, j + d], g[i, j + d]}];
Graphics3D[Table[f[i, j], {i, 0, 2 Pi (n - 1) / n, d}, {j, 0, 2 Pi (n - 1) / n, d}]]
```

Out[136]=



Cubical Type Theory: a constructive interpretation of the univalence axiom*

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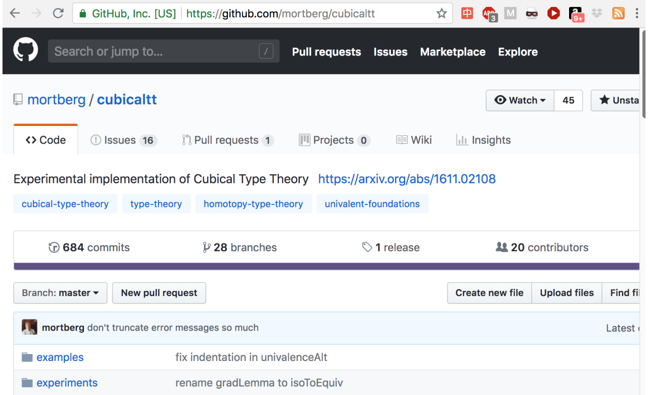
Abstract

This paper presents a type theory in which it is possible to directly manipulate n-dimensional cubes (points, lines, squares, cubes, etc.) based on an interpretation of dependent type theory in a cubical set model. This enables new ways to reason about identity types, for instance, function extensionality is directly provable in the system. Further, Voevodsky's univalence axiom is provable in this system. We also explain an extension with some higher inductive types like the circle and propositional truncation. Finally we provide semantics for this cubical type theory in a constructive meta-theory.

A cubical proof assistant



Fossil hunting at a latitude of 78.3