Set-level mathematics

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Outline

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Definition of set

$$\mathrm{iscontr}(X) := \sum_{x:X} \prod_{y:X} x = y$$

$$\mathrm{isaprop}(X) := \prod_{x,y:X} x = y$$

$$\mathrm{isaset}(X) := \prod_{x,y:X} \mathrm{isaprop}(x = y)$$

A **set** is a type whose path types are all *propositions*.

Definition of h-Levels

$$\begin{aligned}
\mathsf{isofhlevel}(n, X) &: \mathsf{Nat} \to \mathcal{U} \to \mathsf{hProp} \\
\mathsf{isofhlevel}(0, X) &:= \mathsf{iscontr}(X) \\
\mathsf{isofhlevel}(S(n), X) &:= \prod_{x, x' : X} \mathsf{isofhlevel}(n, x = x')
\end{aligned}$$

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Exercise

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Exercise

$$\left(\prod_{X : \mathcal{U}} \mathrm{isaprop}(X) \right) = \mathrm{isofhlevel}(1, X)$$

Hence:

A **set** is a type of hlevel 2.

Topological intuition

Recall:

- Each type X is to be a thought of a *space*
- For two points $a, b : X$, the type $a =_X b$ is the *space of paths* from a to b

Fact (from topology)

A ("nice") space all of whose path spaces are contractible is (homotopy-)equivalent to a discrete space.

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A ("nice") space all of whose path spaces are contractible is (homotopy-)equivalent to a discrete space.

This can be made more precise using the *model in simplicial sets*.

Closure properties

- $\sum_{x:A} B(x)$ is a set if A and all $B(x)$ are
- $A \times B$ is a set if A and B are
- $\prod_{x:A} B(x)$ is a set if all $B(x)$ are
- $A \to B$ is a set if B is

- Any property is a set

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Do you know

- a type that is a set?
- a type for which you don't know (yet) whether it is a set?

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- Most “traditional” mathematics can be done in UniMath with sets (groups, rings, topological spaces, etc.)
- Categories or the type of *all* groups (rings, etc.) have h-level 3.
- Higher category theory and synthetic homotopy theory require higher types.

Outline

Decidable equality

Definition

A type X is **decidable** if we can write a term of type

$$X + \neg X$$

Definition

A type X has **decidable path-equality** if we can write a term of type

$$\prod_{x, x' : A} (x = x') + \neg(x = x')$$

(that is, if all its paths types are decidable)

Hedberg's theorem

Theorem

If a type X has decidable equality, then it is a set.

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Note

Hedberg's theorem is *hard*. There is also an easier proof that Bool and Nat are sets.

Are all types sets?

Is there a type that is not a set?

Great question! It depends:

- In "spartan" type theory some types can't be shown to be sets.
- Assuming univalence, some types can be shown not to be sets.

Another set

Theorem

The type

$$\mathsf{hProp}_{\mathcal{U}} :\equiv \sum_{X:\mathcal{U}} \mathsf{isaprop}(X)$$

is a set.

The proof relies on the univalence axiom for the unviverse $\mathcal{U}$.

Exercise

How would you generalize the above statement to any h-level?
How would you attempt proving it?

Types that are **not** sets

Exercise

Let $\mathcal{U}$ be a univalent universe that contains the type Bool. Why is $\mathcal{U}$ not a set?

Which property of Bool does the proof of the above result exploit?

Outline

Sets and propositions

Types representing properties of sets are usually propositions.

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Example

Given $f : X \to Y$,

$$\mathrm{isInjective}(f) :\equiv \prod_{x,x':X} f(x) = f(x') \to x = x'$$

is not a proposition in general, but it is if X and Y are sets.

Predicates on types

A **subtype** A on a type X is a map

$$A : X \to \mathsf{hProp}_U$$

Exercise

Show that the type of subtypes of any type X is a set.

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The **carrier** of a subtype A is the type of elements satisfying A:

$$\mathrm{carrier}(A) := \sum_{x:X} A(x)$$

Predicates and injections

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Exercise

$$\mathrm{isaset}(X) \to \mathrm{isInjective}(\mathrm{incl}_A)$$

Predicates and injections

Conversely, given a map $f : A \to X$, we can form the function $\chi_f : X \to \mathcal{U}$ given by

$$\chi_f(x) :\equiv \sum_{a:A} f(a) = x$$

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$$\mathsf{isaset}(X) \to \mathsf{isInjective}(f) \to \prod_{x:X} \mathsf{isaprop}(\chi_f(x))$$

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$$\mathsf{isaset}(X) \to \mathsf{isInjective}(f) \to \prod_{x:X} \mathsf{isaprop}(\chi_f(x))$$

(Hard) exercise

ξ_f and incl_A establish an isomorphism between $X \to \mathsf{hProp}_{\mathcal{U}}$ and $\mathsf{injections}(X) :\equiv \sum_{A:\mathcal{U}} \mathsf{isaset}(A) \times \sum_{f:A\to X} \mathsf{isInjective}(f)$

Relations on a type

A **binary relation** R on a type X is a map

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Exercise

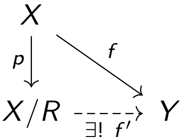
Formulate the properties of being symmetric, transitive, an equivalence relation.

Set-level quotient

Given a set X and relation R on X, the **quotient**

$$X \xrightarrow{p} X/R$$

is defined by the property that any compatible map f **into a set** Y factors uniquely through p:



Exercise

Formulate this condition precisely.

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First we define for a subtype $A : X \to \mathrm{hProp}_U$

$$\begin{aligned}\mathrm{iseqclass}(A) := &\ \|\mathrm{carrier}(A)\| \\ &\times \prod_{x,y:A} Rxy \to Ax \to Ay \\ &\times \prod_{x,y:A} Ax \to Ay \to Rxy\end{aligned}$$

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Then we define

$$X/R := \sum_{A:X \to \mathrm{hProp}_U} \mathrm{iseqclass}(A)$$

Outline

Paths between pairs

Given $B : A \to \mathcal{U}$ and $a, a' : A$ and $b : B(a)$ and $b' : B(a')$,

$$(a, b) = (a', b') \quad \simeq \quad \sum_{p : a = a'} \mathsf{transport}^B\Big(p,\ b\Big) = b'$$

If $B(x)$ is a proposition for any $x : A$, then this simplifies to

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Exercise

Why?

Example: natural numbers

An **even natural number** is a pair consisting of a natural number and a proof of its evenness.

$$\mathsf{iseven}(n) :\equiv \sum_{k:\mathsf{Nat}} k + k = n \qquad \mathsf{evennat} :\equiv \sum_{n:\mathsf{Nat}} \mathsf{iseven}(n)$$

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The type iseven(n) hence should be a proposition.

Exercise

It is!

Groups

Traditionally (in set theory), a group is a quadruple (G, m, e, i) of

- a set G
- a multiplication $m : G \times G \to G$
- a unit $e \in G$
- an inverse $i : G \to G$

subject to the usual axioms.

Groups in type theory

In type theory, a group is a (dependent) pair $(data, proof)$ where

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This is in turn guaranteed as long as the underlying type G is required to be a *set*.

Exercise

Why?

Group isomorphisms

The type of groups is

$$\mathsf{Grp} := \sum_{X:\mathsf{hSet}} \mathsf{GrpStructure}(X)$$

A **group isomorphism** $G \to G'$ is

- a bijective function on the underlying sets $X \to X'$
- compatible with the group structures S and S' on X and X'.

Identity is isomorphism for groups

$$
\begin{aligned}
G = G' &\simeq (X, S) = (X', S') \\
&\simeq \sum_{p : X = X'} \mathrm{transport}^{\mathrm{GrpStructure}}(p, S) = S' \\
&\simeq \sum_{p : X = X'} (\mathrm{transport}^{Y \mapsto (Y \times Y \to Y)}(p, m) = m') \\
&\qquad\quad \times (\mathrm{transport}^{Y \mapsto (Y \to Y)}(p, i) = i') \\
&\qquad\quad \times (\mathrm{transport}^{Y \mapsto (1 \to Y)}(p, e) = e') \\
&\simeq \sum_{f : X \simeq X'} \left(f \circ m \circ (f^{-1} \times f^{-1}) = m'\right) \\
&\qquad\quad \times \left(f \circ i \circ f^{-1} = i'\right) \\
&\qquad\quad \times (f \circ e = e') \\
&\simeq (G \cong G')
\end{aligned}
$$